CONTINUOUS FUNCTIONS

DEFINITION: Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces. A function $F : X \to Y$ is called:

- **continuous** iff pre-images of open sets are open. That is, $F^{-1}(U) \in \mathcal{T}$ for all $U \in \mathcal{U}$.
- **open** iff images of open sets are open. That is, $F(T) \in \mathcal{U}$ for all $T \in \mathcal{T}$.
- **closed** iff images of closed sets are closed. That is, if $X \backslash B \in \mathcal{T}$, then $Y \backslash F(B) \in \mathcal{U}$.

EXAMPLE: Let $X = \{1, 2\}$ and $Y = \{a, b, c\}$. Let $\mathcal{U} = \{\emptyset, \{a\}, \{a, b\}, Y\}$.

- Let $\mathcal{T}$ be the discrete topology on X. Show $F = \{(1, a), (2, b)\}$ is continuous but neither open nor closed.

- Let $\mathcal{T}$ be the indiscrete topology on X. Show $F = \{(1, a), (2, b)\}$ is open but neither closed nor continuous.

- Let $\mathcal{T}$ be the indiscrete topology on X. Find a function F which is closed but neither open nor contiuous.

EXAMPLE: Consider $\mathbb{R}$ with the Euclidean Topology.

Show the definition of **continuous** from Calculus matches this 'new' definition of continuous.

EXAMPLE: Let the **Heaviside Function** is defined as: $H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$.

Let $X = \mathbb{R}$ and let $\mathcal{T}$ be the Euclidean topology on $\mathbb{R}$ and let $\mathcal{S}$ be the Sorgenfrey topology on $\mathbb{R}$.

Which of the following functions are continuous?

- $H : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T})$

- $H : (\mathbb{R}, \mathcal{S}) \to (\mathbb{R}, \mathcal{S})$

- $H : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{S})$

- $H : (\mathbb{R}, \mathcal{S}) \to (\mathbb{R}, \mathcal{T})$

EXAMPLE: Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces and $F : X \to Y$.

- Prove if $\mathcal{T}$ is the discrete topology or if $\mathcal{U}$ is the indiscrete topology, then F is continuous.

- Prove F is continuous iff pre-images of closed sets are closed.

DEFINITION: Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X.

If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say $\mathcal{T}_1$ is **coarser** (or **weaker**) than $\mathcal{T}_2$ and $\mathcal{T}_2$ is **finer** (or **stronger**) than $\mathcal{T}_1$.

QUESTION: What is the coarsest topology we can put on a set? The finest?

EXAMPLE: Show the Sorgenfrey topology on $\mathbb{R}$ is finer than the Euclidean topology on $\mathbb{R}$.

EXAMPLE: Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X with $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Let $\mathcal{U}$ be a topology on Y.

Prove if $F : (X, \mathcal{T}_1) \to (Y, \mathcal{U})$ is continuous, then so is $F : (X, \mathcal{T}_2) \to (Y, \mathcal{U})$.

EXAMPLE Suppose $F : X \to Y$. Let $\mathcal{T}_F = \left\{ F^{-1}(U) \ : \ U \in \mathcal{U} \right\}$.

- Show $\mathcal{T}_F$ is a topology.

- Show $\mathcal{T}_F$ is the coarsest topology on X for which F is continuous.